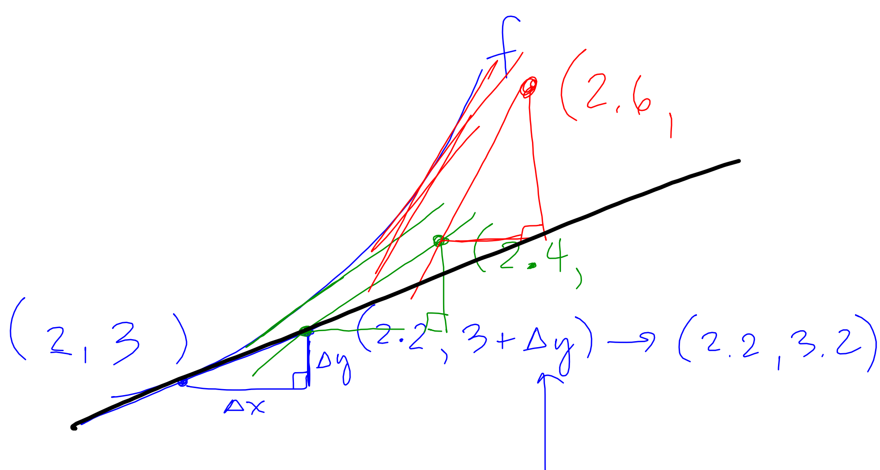
Calculus Warm Up # 6-4

1. Let f(x) be the solution to the differential equation with initial condition f(2) = 3. Use Euler's method to approximate f(2.6) with 3 equal steps from x = 2.

$$\frac{dy}{dx} = 2x - y$$

2. Find the first 4 non-zero terms of the Maclaurin series to approximate f(x) and use it to approximate f(1.5).

$$f(x) = \frac{e^x + e^{-x}}{2}$$



$$(\Delta x)\frac{dy}{dx} = \frac{\Delta y}{\cancel{\Delta x}}\cancel{\Delta x}$$

From Monday:

10.8 Power Series

Determining Domain

Radius of Convergence

Interval of Convergence

Power Series centered at x = 0 or x = c

(c = constant)

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n + \ldots$$

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \ldots + a_n(x-c)^n + \ldots$$

*To simplify notation, it is agreed that $(x-c)^0 = 1$, even if $x = c$.

$0^0 = 1$

Domain of a Power Series

1) A single point (the center)
2) An interval centered at c
3) All reals

Radius of Convergence, R.

(distance from the center)

1) Series converges only at c: R = 0
2) If there is a real number, R > 0 such that for $|x - c| < R$, the series converges absolutely, $|x - c| > R$, it diverges.
3) Series converges for all reals, $R = \infty$

Find the radius of convergence:

Ex: $\sum_{n=0}^{\infty} 3(x-2)^n$

centered at x = 2,

Ratio Test for $x \neq 2$

$$\lim_{n\to\infty}\left|\frac{3(x-2)^{n+1}}{3(x-2)^n}\right| = \lim_{n\to\infty}|x-2| = |x-2|$$

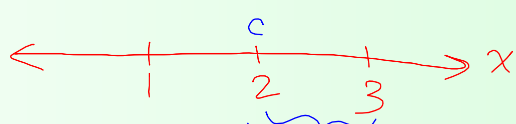
$\therefore$ Series converges for $|x-2|<1$

and diverges for $|x-2|>1$

$|x-2|<1$

$-1 < x-2 < 1$

$1 < x < 3$



$\boxed{R=1}$

Finding an interval of convergence:

Ex: $\sum_{n=1}^{\infty} \frac{x^n}{n}$ centered at x = 0,

Ratio Test: $\lim_{n\to\infty} \left| \frac{x^{(n+1)}}{(n+1)} \cdot \frac{n}{x^n} \right|$

$= \lim_{n\to\infty} \left| \frac{n}{n+1} \right| |x| = |x|$

$\boxed{R=1}$

interval of convergence: $(-1, 1)$, but we need to consider the endpts...

@ $x=1$

$\sum_{n=1}^{\infty} \frac{1}{n}$ divergent harmonic p-series

@ $x=-1$

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} \cdots$

converges.

$\therefore$ Interval of convergence $= [-1, 1)$

One more:

$$\sum_{n=0}^{\infty} \frac{(-1)^n (x)^{2n+1}}{(2n+1)!}$$

$\lim_{n\to\infty} \left| \frac{x^{2(n+1)+1}}{(2(n+1)+1)!} \cdot \frac{(2n+1)!}{x^{2n+1}} \right|$

$= \lim_{n\to\infty} \left| \frac{(2n+1)!}{(2n+3)!} \cdot \frac{x^{2n+3}}{x^{2n+1}} \right|$

$= \lim_{n\to\infty} \left| \frac{x^2}{(2n+3)(2n+2)} \right|$

$= 0$ for all x

so $R = \infty$

Today:

Properties of a function defined by a Power Series:

$$f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \ldots + a_n(x-c)^n + \ldots$$

If $R > 0$, then on the interval: $(c - R, c + R)$

$f(x)$ is continuous, differentiable and integrable.

Both differentiation and Integration can be taken term by term.

The radius of convergence for a Power Series will be the same for the functions obtained by differentiating and/or integrating.

The interval of convergence for a Power Series <u>may differ</u> for the functions obtained by differentiating and/or integrating, because of endpoint behavior.

Let's investigate:

Find the intervals of convergence for

f(x), f′(x), and $\int f(x)\,dx$

$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots$$

$$\lim_{n\to\infty} \left| \frac{x^{n+1}}{n+1} \cdot \frac{n}{x^n} \right|$$

$$= \underbrace{\lim_{n\to\infty} \left| \frac{n}{n+1} \right|}_{1} \cdot |x|$$

converges when

$|x| < 1$
$-1 < x < 1$ $\Big\} R = 1$

endpts

@ $x = 1$: $\sum_{n=1}^{\infty} \frac{1}{n}$ divergent p-series

@ $x = -1$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ convergent alternating p-series

interval of convergence for f(x) $\Big\}$ $[-1, 1)$

$$f'(x) = \sum_{n=1}^{\infty} x^{n-1} = 1 + x + x^2 + x^3 + \ldots$$

@ $x = \pm 1 \rightarrow \sum_{n=1}^{\infty} (\pm 1)^{n-1}$ ← $|r| = 1$, diverges

$f'(x) \rightarrow$ interval of convergence: $(-1, 1)$

$$\int f(x)\,dx = C + \sum_{n=1}^{\infty} \frac{x^{n+1}}{n(n+1)} = C + \frac{x^2}{1\cdot 2} + \frac{x^3}{2\cdot 3} + \frac{x^4}{3\cdot 4} + \ldots$$

@ $x = \pm 1 \rightarrow \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

interval: $[-1, 1]$

compare to $\sum \frac{1}{n^2}$

p-series $p = 2$ $\Big\}$ converges

HW: p. 623, 17 - 25, skip 21,

and #31, 33

green #2 wait

Classwork:

2009 FR practice # 1 - 6

This will carry over into next week.

*Answers to FR questions will be posted this afternoon.

Yellow BC FR Practice

6b) $f(0)=1$

$f'(0)=-2\sin(0)=0$

$f''(0)=-4\cos(0)=-4$

$f'''(0)=0$

$f^{(4)}(0)=16\cos(0)=16$

$f^{(5)}(0)=0$

$f^{(6)}(0)=-64$

$$P(x)=1-\frac{4x^2}{2!}+\frac{16x^4}{4!}-\frac{64x^6}{6!}$$

general term

$$a_n=\frac{(-1)^n 2^n x^{2n}}{(2n)!}$$

Yellow

6c) $g(x) = \frac{P(x)}{x^2}$

★ the subtract 1 on top goes away b/c you are differentiating

$$g(x) = -\frac{4}{2!} + \frac{16x^2}{4!} - \frac{64x^4}{6!} + \cdots + \underbrace{\frac{(-1)^n 2^{2n} x^{2n-2}}{(2n)!}}_{\text{general term}}$$

d) $g'(x) = \frac{32x}{24} - \frac{4(64x^3)}{6!} + \cdots$

$g'(0) = 0$

$g''(x) = \frac{32}{24} - \frac{12(64x^2)}{6!} + \cdots$

$g''(0) = \frac{32}{24}$ +, so min @ $x = 0$